Physics in low dimensions and cold atoms

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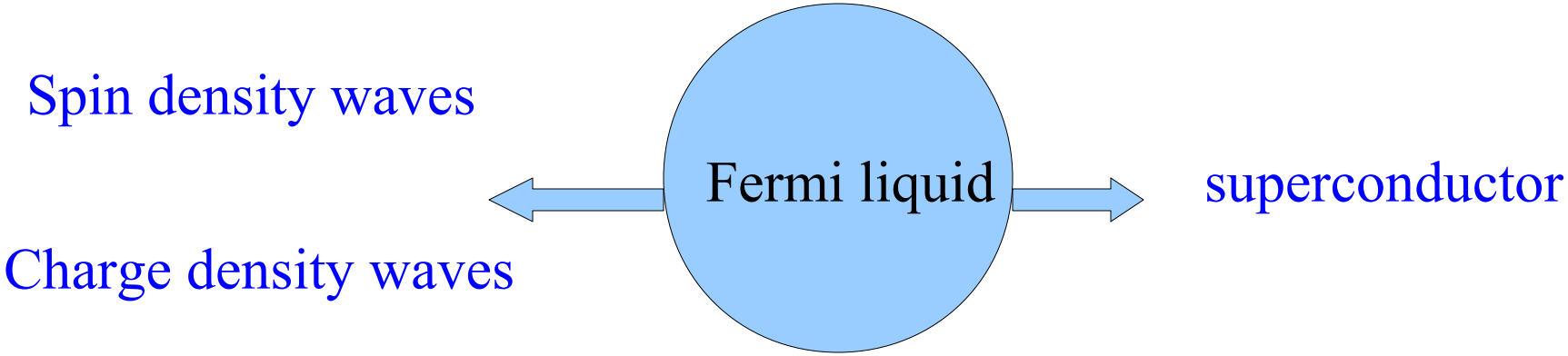
Outline

- The 1D case : instabilities and the Luttinger liquid concept
- The Renormalization Group at work on sine-Gordon theory
- Multicomponent case : a interesting example
- The 2D case : a new state of matter, the FQHE
- Abelian states and non-Abelian states
- Luttinger liquids again from edge combination(s)

- Conclusions and outlook

Instabilities in 3D

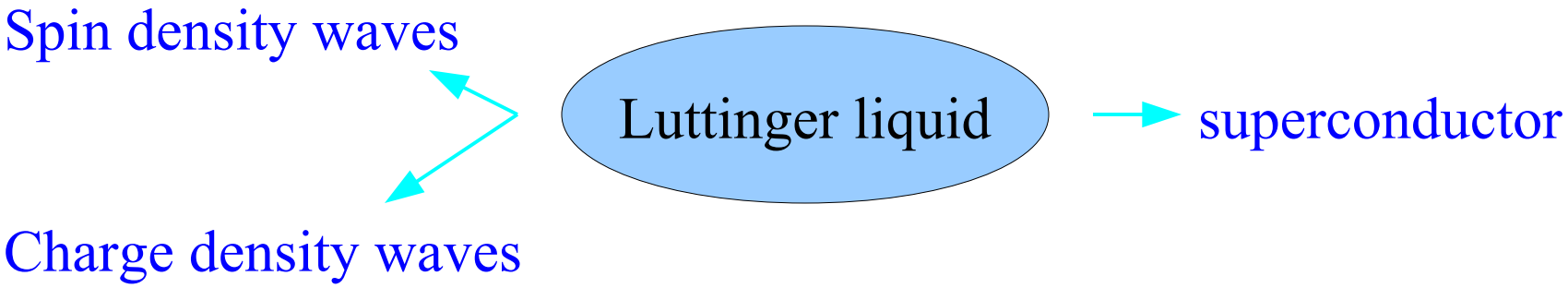
True long-range order is allowed



One-dimensional world

LRO becomes Quasi-LRO

Algebraic decay of correlation functions



Criticality in 1D

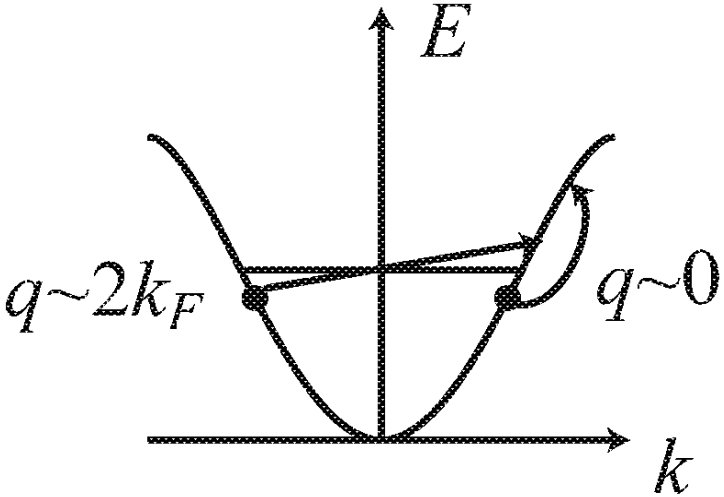
In 1D RG is very powerful: several simple fixed points

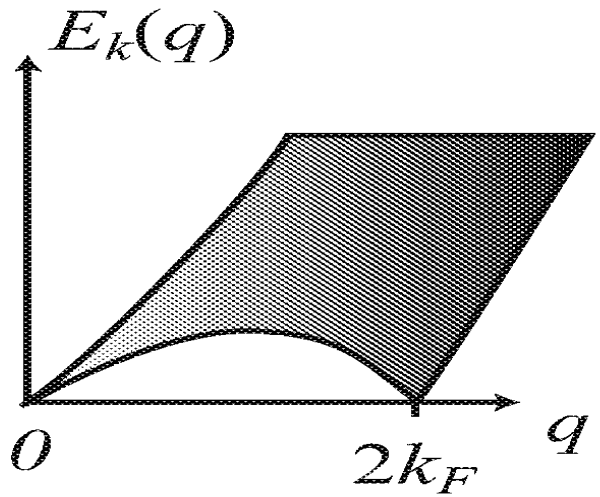
A free massless boson field theory is the fixed point
For fermions and bosons in 1D: “Luttinger liquid”

$$\mathcal{L} = \int dx \frac{1}{2K} \left[(\partial_t \phi)^2 - (\partial_x \phi)^2 \right]$$

Observables are not the field itself but gradients and
Exponentials...hence the scale K

Effective low-energy theory





Two-component mixtures: use pseudo-spin notation σ= ✸, ❋

$$\hat{n}_\sigma(x) \sim (n_\sigma - \partial_x \phi_\sigma/\pi) \sum_s e^{2is(\pi n_\sigma x - \phi_\sigma)}$$

$$\Psi^\dagger_\sigma(x) \sim (n_\sigma - \partial_x \phi_\sigma/\pi)^{1/2} \sum_s e^{is(\pi n_\sigma x - \phi_\sigma)} e^{-i\theta_\sigma}$$

sine-Gordon RG flow

$$\mathcal{L} = \int dx \frac{1}{2K} \left[(\partial_t \phi)^2 - (\partial_x \phi)^2 \right] + y \int dx \cos(\sqrt{4\pi}\phi)$$

$$\frac{dy}{d\ell} = (2 - K)y$$

$$\frac{dK}{d\ell} = -y^2$$

$$\Lambda(\ell) = \Lambda \mathrm{e}^{-\ell}$$

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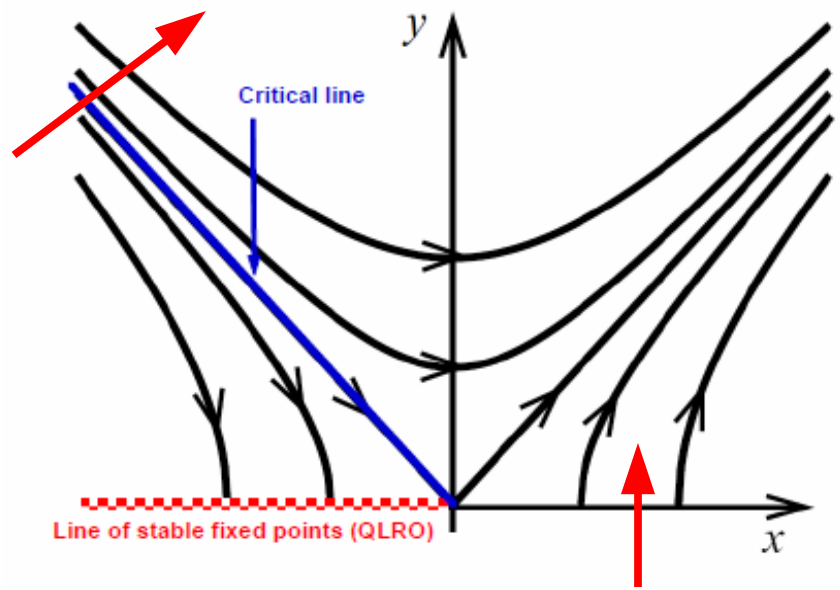
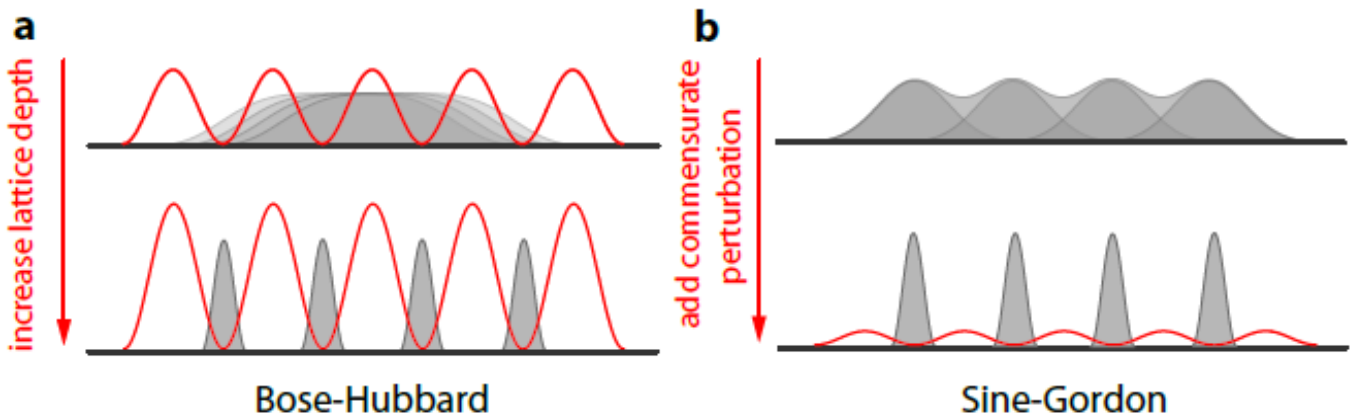


K*=2

sine-Gordon type phase transitions



The 1D Bose is a Luttinger liquid with K given by
The scattering length

Haller et al., nature 2010

Two-component system

Effect of commensurability in a simple many-body problem

E. Burovsky, G. Orso, Th. Jolicoeur, PRL 2009
G. Orso, E. Burovsky, Th. Jolicoeur, PRL 2010, CRAS 2010
G. Roux, E. Burovsky, Th. Jolicoeur, PRA 2011

Effective low-energy theory

$$\mathcal{H}_{\text{free}} = \mathcal{H}_0(\phi_\uparrow) + \mathcal{H}_0(\phi_\downarrow)$$

$$\mathcal{H}_0(\phi_\sigma) = \frac{v_\sigma}{2\pi} \int dx \left[K_\sigma (\pi \Pi_\sigma)^2 + K_\sigma^{-1} \left(\partial_x \phi_\sigma\right)^2 \right]$$

Non-interacting fermions: $K_\sigma = 1$

<u>Effect of interactions:</u>

$$\mathcal{H}_{\text{int}} = \int dx dx' \, U_{\sigma\sigma'}(x - x') \hat{n}_\sigma(x) \hat{n}'_\sigma(x')$$

- $K_\sigma \neq 1$
- $\mathcal{H}_1 = g \int dx \, (\partial_x \phi_\uparrow)(\partial_x \phi_\downarrow)$
- higher harmonics

The effects of higher harmonics

$$\mathcal{H}_h = \sum_{s,s'>0} G_{ss'} \int dx \, \cos\left[2(sk_F^{\uparrow} - s'k_F^{\downarrow})x - 2(s\phi_{\uparrow} - s'\phi_{\downarrow})\right]$$
$$+ \sum_{s,s'>0} \tilde{G}_{ss'} \int dx \, \cos\left[2(sk_F^{\uparrow} + s'k_F^{\downarrow})x - 2(s\phi_{\uparrow} + s'\phi_{\downarrow})\right]$$

$$p\, n_{\uparrow} - q n_{\downarrow} = 0$$

(*p* and *q* are integers)

$$\mathcal{H}_2 = G \int dx \, \cos 2\left(p\, \phi_{\uparrow}(x) - q\phi_{\downarrow}(x)\right)$$

p = *q* = 1 ⇓ spin gap (attractive interactions)

The massive phase

Consider a special case: $v_\uparrow = v_\downarrow \qquad g = 0$

The exact eigenmodes are:

$$\begin{cases} \phi_a = (p\phi_\uparrow - q\phi_\downarrow)/\sqrt{2}\,, & \text{(massive)} \\ \phi_b = (qK_\downarrow\phi_\uparrow + pK_\uparrow\phi_\downarrow)/\sqrt{2} & \text{(massless)} \end{cases}$$

The corresponding Luttinger parameters:

$$\begin{cases} K_a = \left(p^2K_\uparrow + q^2K_\downarrow\right)/2 \\ K_b = K_aK_\uparrow K_\downarrow\ . \end{cases}$$

Hence, a sufficient condition: $\begin{cases} v_\uparrow = v_\downarrow \\ p^2K_\uparrow + q^2K_\downarrow \leqslant 2 \end{cases}$

Notice the strong asymmetry between ✸ and ✵
…and a need for finite interactions

Mass-asymetric fluids: 1D Hubbard model

$$H_{\mathrm{aH}} = -\sum_{\langle ij\rangle\sigma} t_\sigma \left(c^\dagger_{i,\sigma} c_{j,\sigma} + h.c.\right) - |U| \sum_i \hat{n}_{i\uparrow}\hat{n}_{i\downarrow}$$

- $t_\downarrow = t_\uparrow$: Bethe-Ansatz solvable even in the polarized case (Penc and Woynarovich, 1975) ; features fully paired ("BCS") and partially polarized ("FFLO")
- $t_\downarrow = 0$: reduces to the spinless Falicov-Kimball model. Has a devil's staircaise of periodic phases (e.g. Gruber et al., 1994, Micheletti et al., 1997)
- $0 < t_\downarrow < t_\uparrow$: has been studied numerically by Batrouni et al., 2009 (QMC), and by Wang et al., 2009 (iTEBD)

Has been studied quite extensively, it seems. Is there anything non-trivial left?

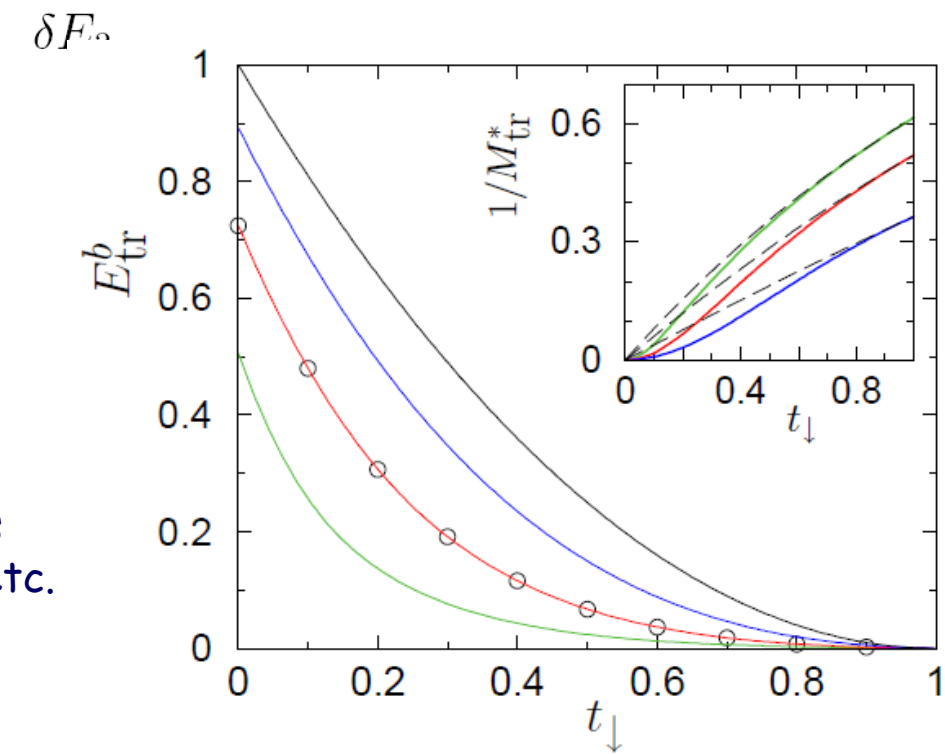
1D Hubbard model: few-body problem

$$H_{\mathrm{aH}} = -\sum_{\langle ij\rangle\sigma} t_\sigma \left(c^\dagger_{i,\sigma} c_{j,\sigma} + h.c.\right) - |U| \sum_i \hat{n}_{i\uparrow}\hat{n}_{i\downarrow}$$

- unequal hoppings: three-body bound states exist in vacuum (e.g., Mattis, 1986)

$$-E = E^b_{\mathrm{pair}} + E^b_{\mathrm{tr}}$$

pair energy



In fact, there exist not only 3-body states (1+2), but also 4-body (1+3), 5-body (2+3) etc.

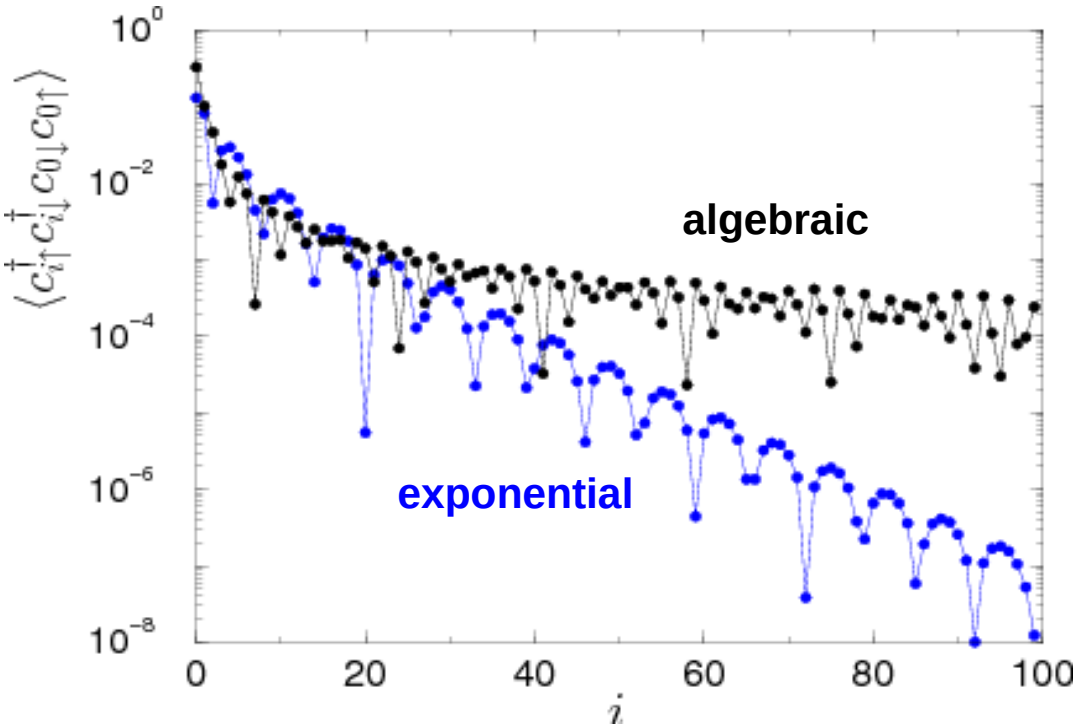
What about many-body physics?

Mass-asymetric fluids: 1D Hubbard model

$$H_{\mathrm{aH}} = -\sum_{\langle ij\rangle\sigma} t_\sigma \left(c^\dagger_{i,\sigma} c_{j,\sigma} + h.c.\right) - |U| \sum_i \hat{n}_{i\uparrow}\hat{n}_{i\downarrow}$$

$$n_\downarrow = 2n_\uparrow$$

$$t_\downarrow / t_\uparrow = 0.3$$



For some parameter values correlations decay exponentially. Why is that?

Quasi long range order

In 1D no true long-range order is possible ⇓ algebraic correlations at most:

$$\langle \mathcal{O}(0)\mathcal{O}^\dagger(x)\rangle \sim \left(\frac{1}{x}\right)^\alpha \quad \Downarrow \quad \chi(\omega) \sim \omega^{\alpha-2}$$

i.e. the slowest decay ⇓ the dominant instability.

Equal densities ($p = q = 1$), attractive interactions :

- $\mathcal{O}_{\mathrm{CDW}} = \psi^\dagger_{L\uparrow}\psi_{R\downarrow} + \mathrm{h.c.}$
- $\mathcal{O}_{\mathrm{SS}} = \psi^\dagger_{L\uparrow}\psi^\dagger_{R\downarrow} + \mathrm{h.c.}$

Unequal densities (e.g. $p = 2$, $q = 1$) :

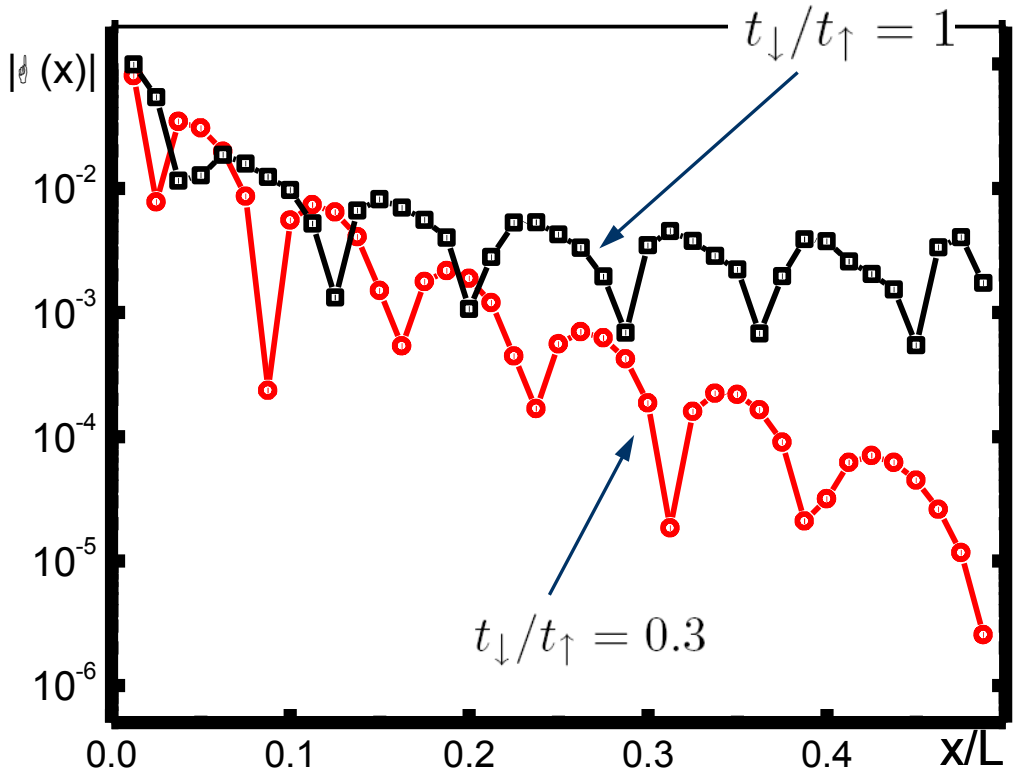
- CDW/ SDW-z correlations are algebraic
- SS correlations are destroyed (i.e. decay exponentially)
- $\mathcal{O}_{2+1} = \Psi^\dagger_\downarrow \Psi^\dagger_\downarrow \Psi^\dagger_\uparrow$ "trimer" ordering

The asymmetric Hubbard model, correlations

unequal hoppings: the model is no longer integrable, hence use DMRG

$$\Gamma(x) = \langle \mathcal{P}_{L/2} \mathcal{P}^{\dagger}_{L/2+x} \rangle \qquad \mathcal{P}_j = c_{j\uparrow} c_{j\downarrow}$$

superconducting correlations



'commensurate' densities

$$n_\downarrow = 2n_\uparrow = 3/10$$

Majority of the heavy species: YES
Majority of the light species: NO

The asymmetric Hubbard model, correlations

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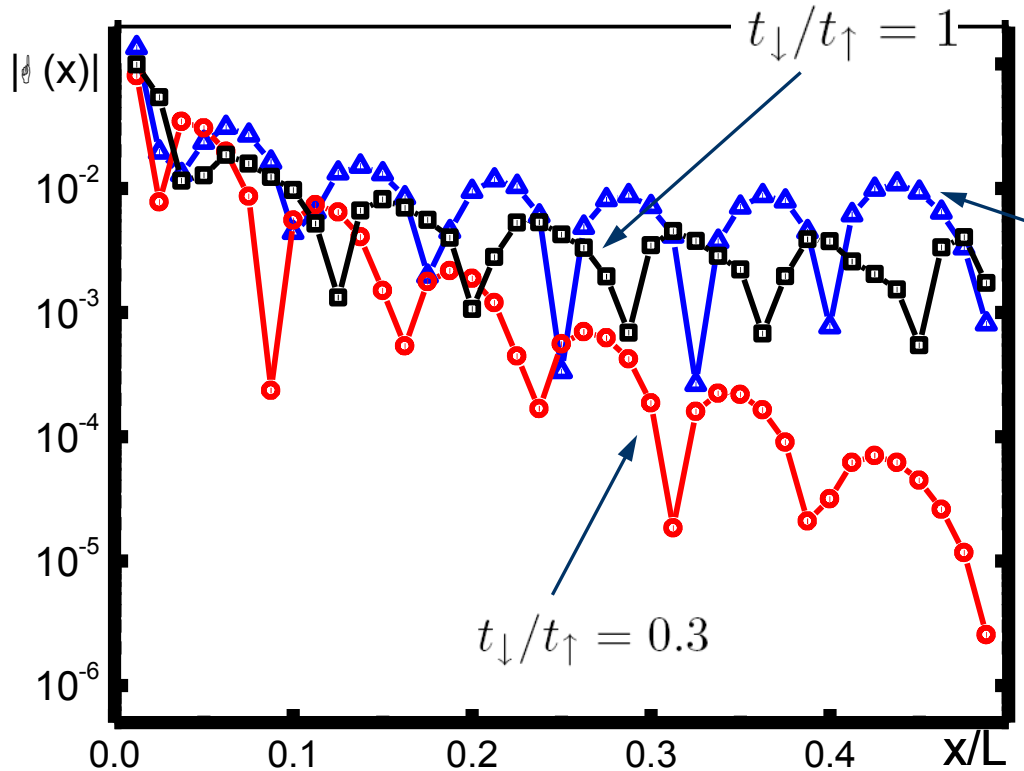
superconducting correlations



'commensurate' densities

$$n_\downarrow = 2n_\uparrow = 3/10$$

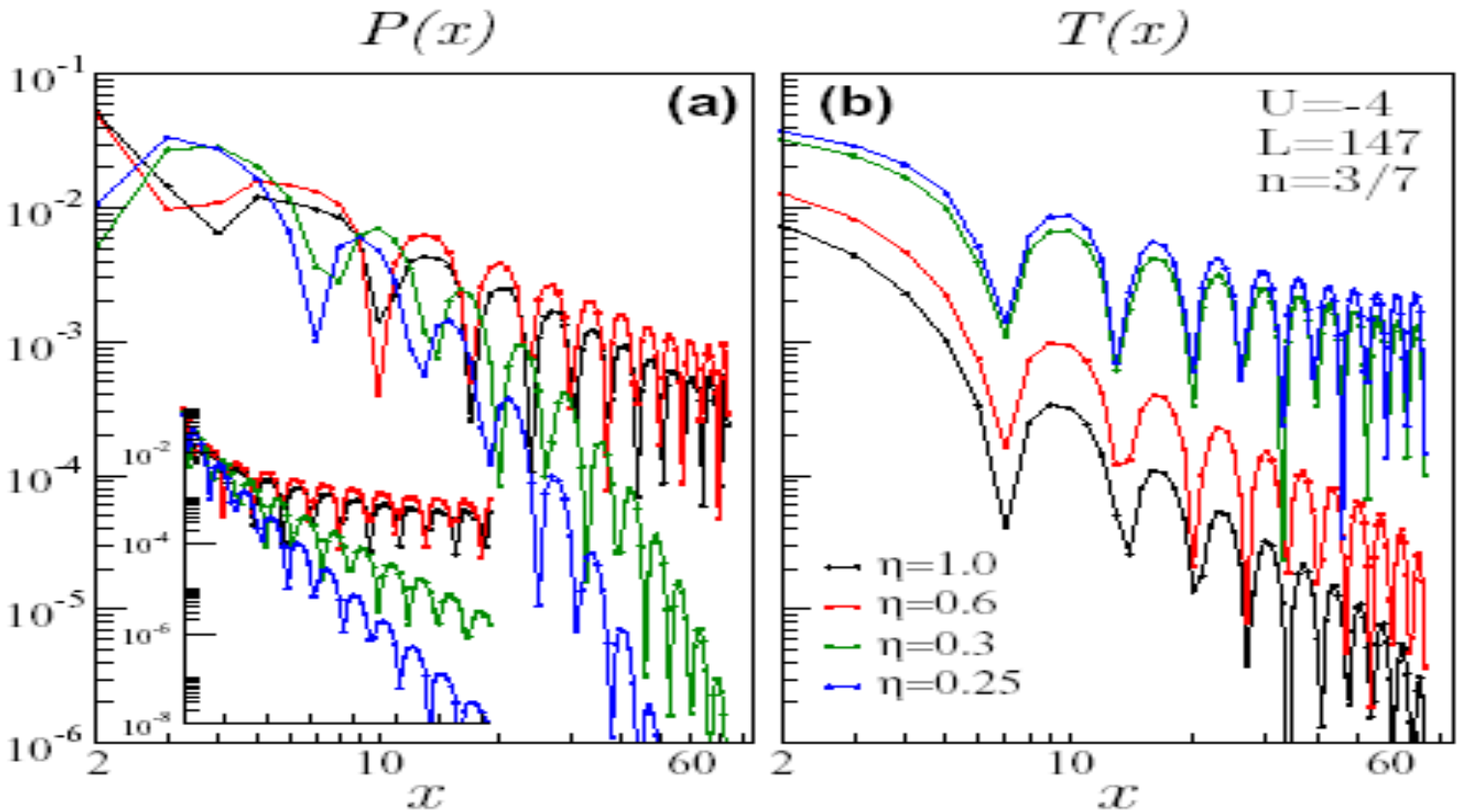
'incommensurate' densities

$$n_\uparrow = 17/80 \quad n_\downarrow = 29/80$$

$$t_\downarrow/t_\uparrow = 0.3$$

Majority of the heavy species: YES
Majority of the light species: NO

Trimer correlations



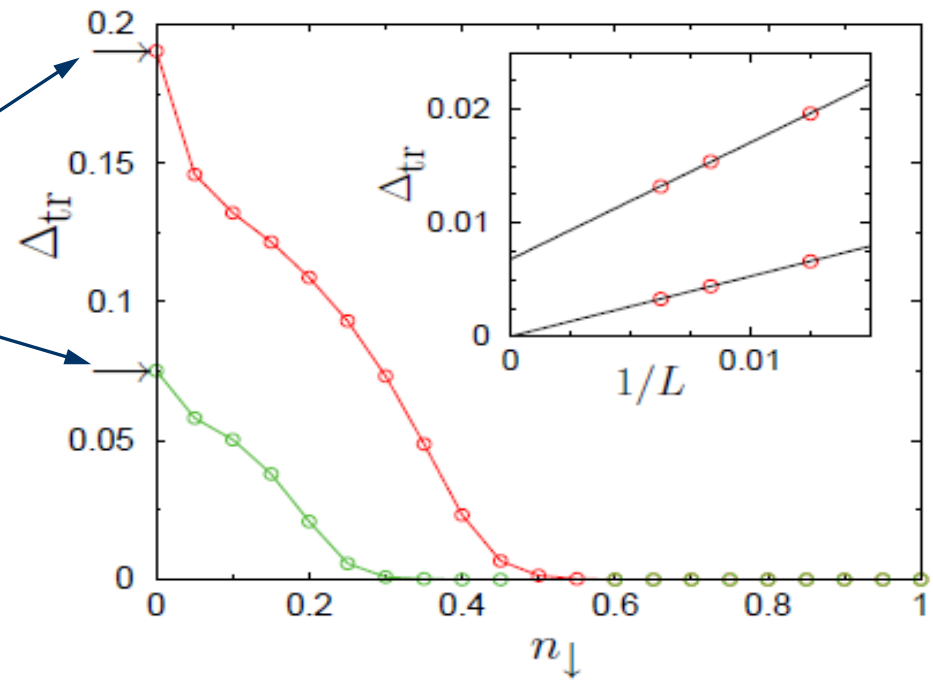
The asymmetric Hubbard model, cont'd

Directly compute the gap:

$$\Delta_{\rm tr} = -\lim_{L\to\infty} E_L(N_\uparrow+1, N_\downarrow+2) + E_L(N_\uparrow, N_\downarrow) - E_L(N_\uparrow+1, N_\downarrow+1) - E_L(N_\uparrow, N_\downarrow+1)\ ,$$

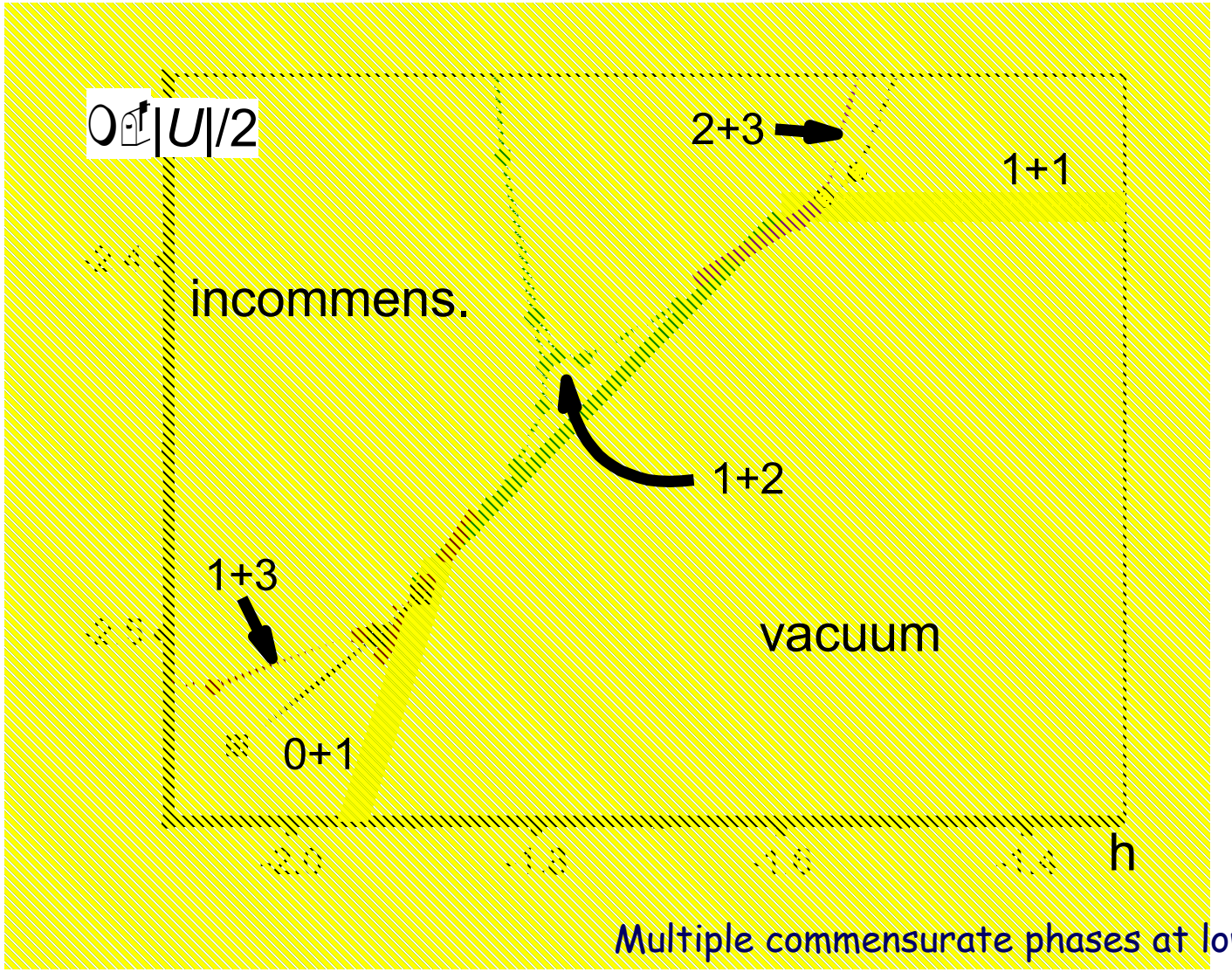
$$n_\downarrow = 2n_\uparrow$$

Single trimer binding energy



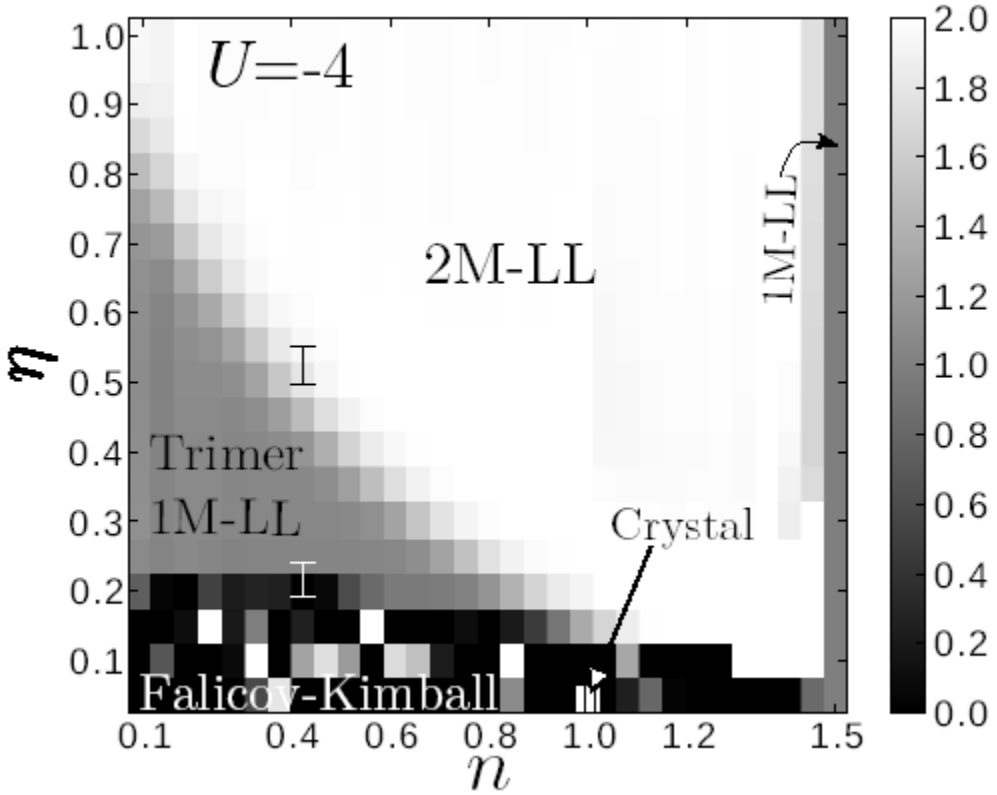
Gap is zero --- algebraic correlations
Gap is nonzero --- exponential correlations

The asym. Hubbard model, phase diagram

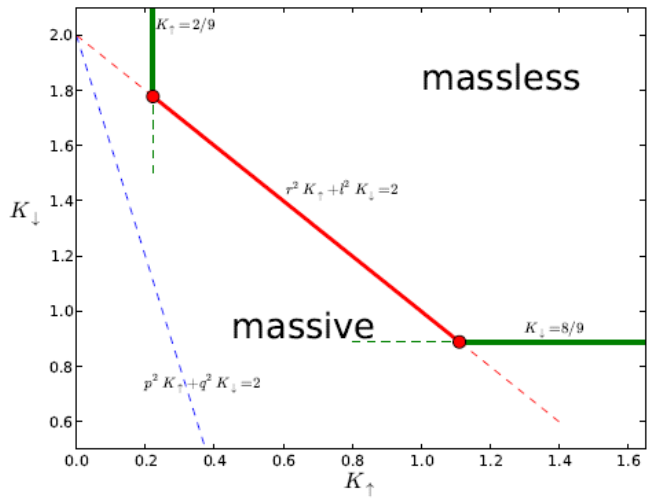


Multiple commensurate phases at low density

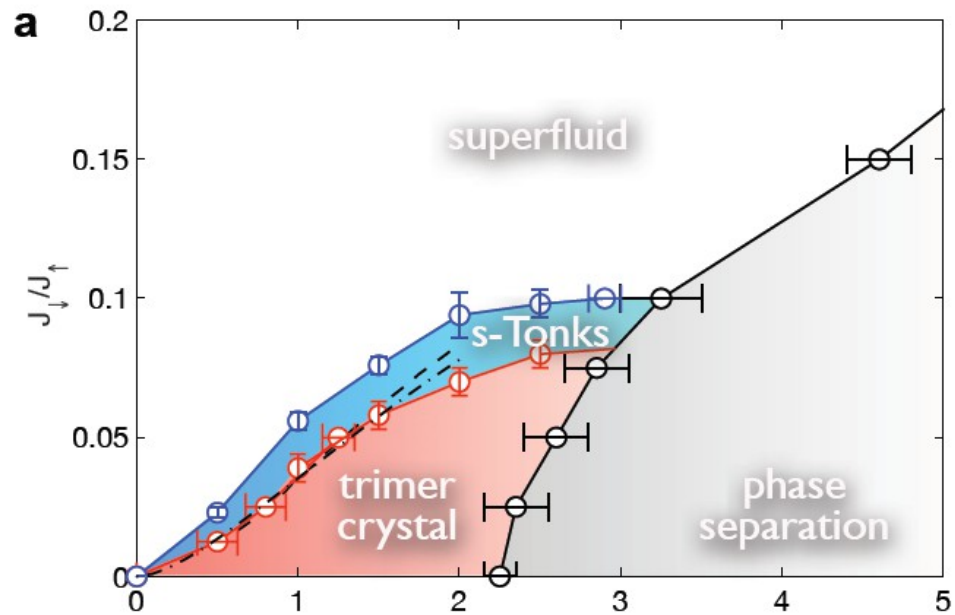
Phase diagram



Lattice effects and "trimer crystal"



Keilmann, Cirac, and Roscilde, 2009 :
"trimer crystal"



The 2D case: new states of matter

Borrowed from Laughlin paper about FQHE

In 2D in a magnetic field: new incompressible liquids
For series of magic rational fillings p/q of the LLL

Quasiparticles with fractional charge and fractional statistics

The bulk is featureless and the QH and QP are dips/bumps
In the density...

Edge state study on the cylinder

Two opposite limits: thin torus vs hoop

Thin torus is electrostatics with a devil staircase

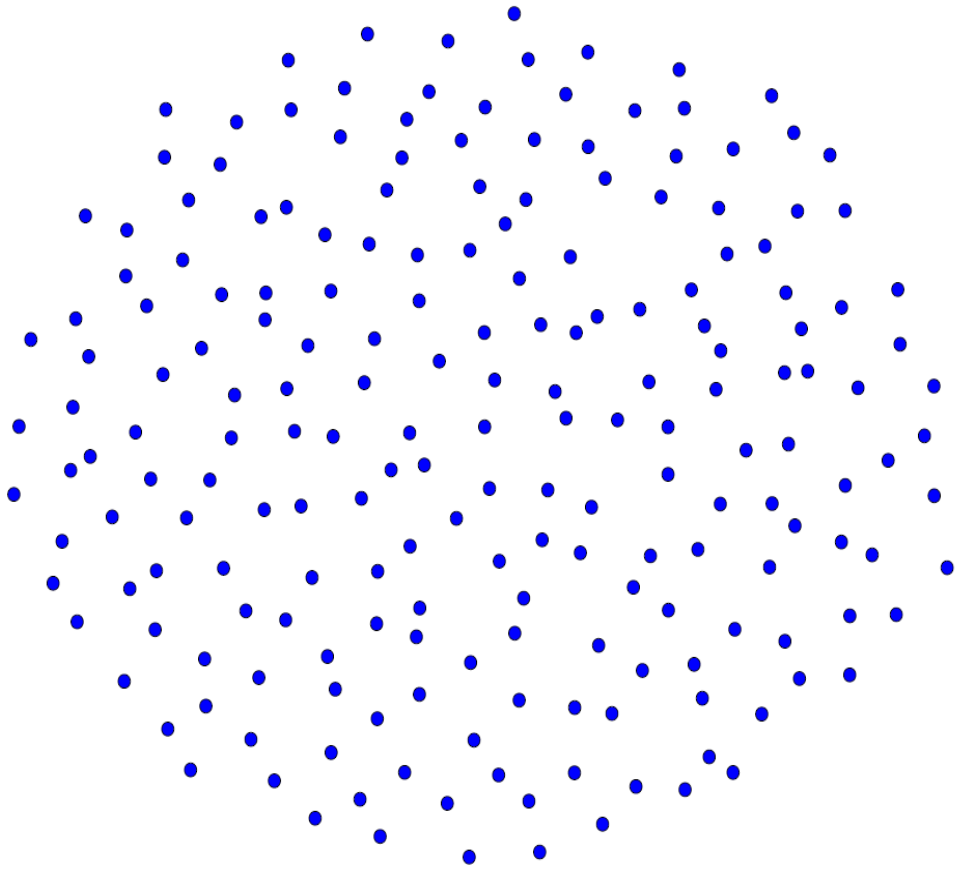
Hoop is Luttinger-like: edges only

Inbetween are composite fermions = particle + vortex
electron+two vortices, boson + one vortex

Work with Paul Soulé

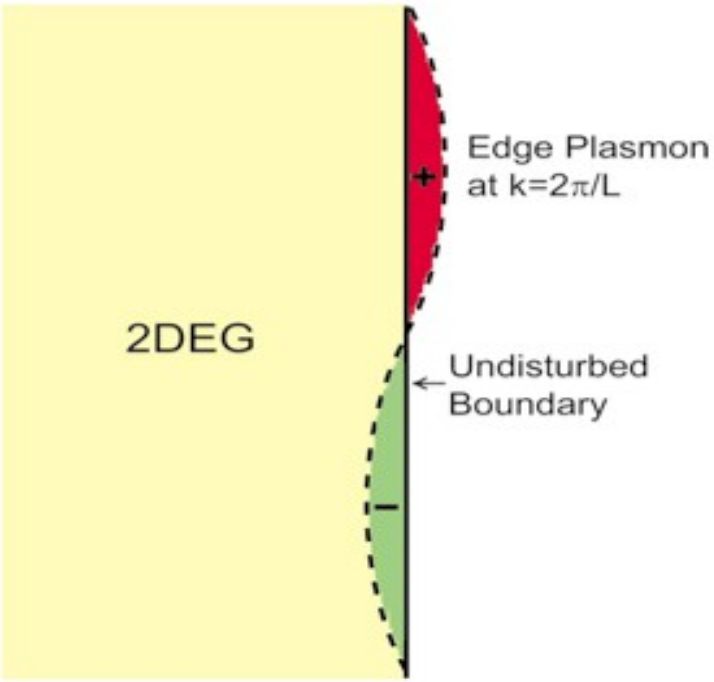
Snapshot in the Laughlin wavefunction

$$\Psi^m_{Laughlin} = \prod_{j<k} (z_j - z_k)^m \prod_j e^{-|z_j|^2/4}$$



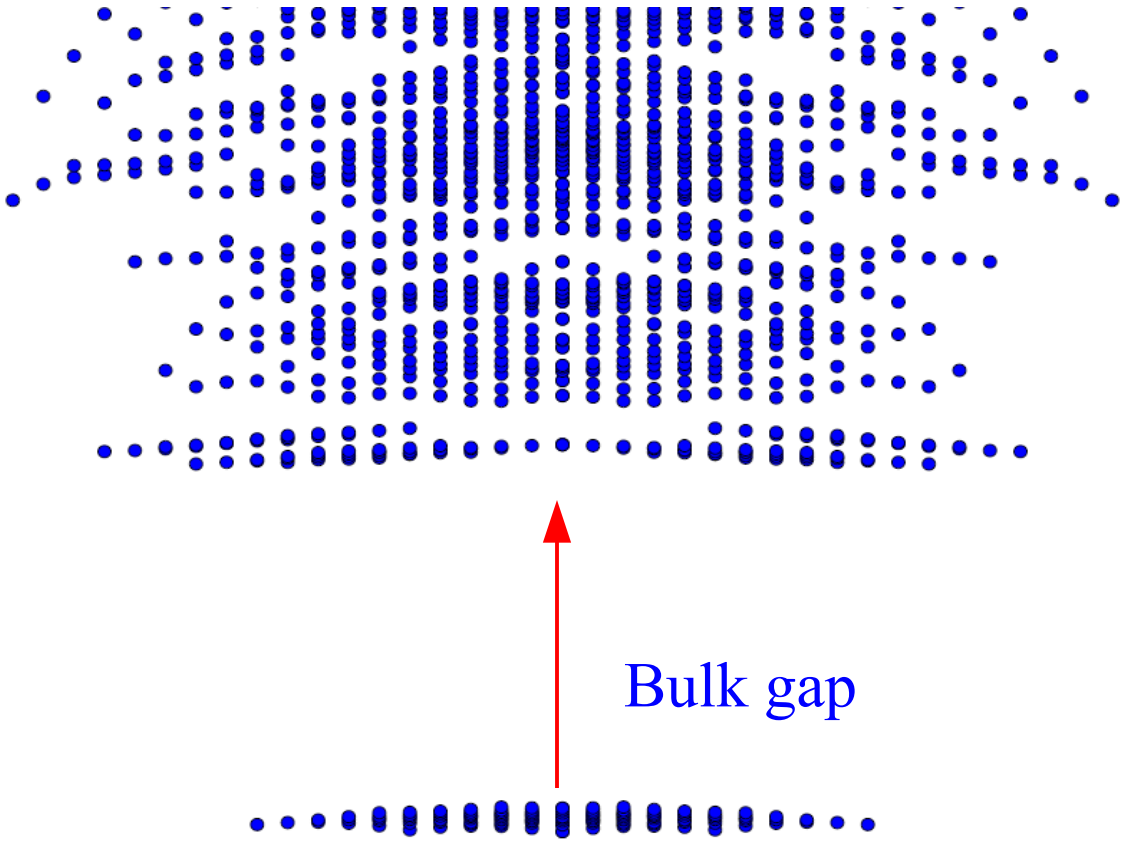
200 electrons at filling factor 1/3

Edge modes

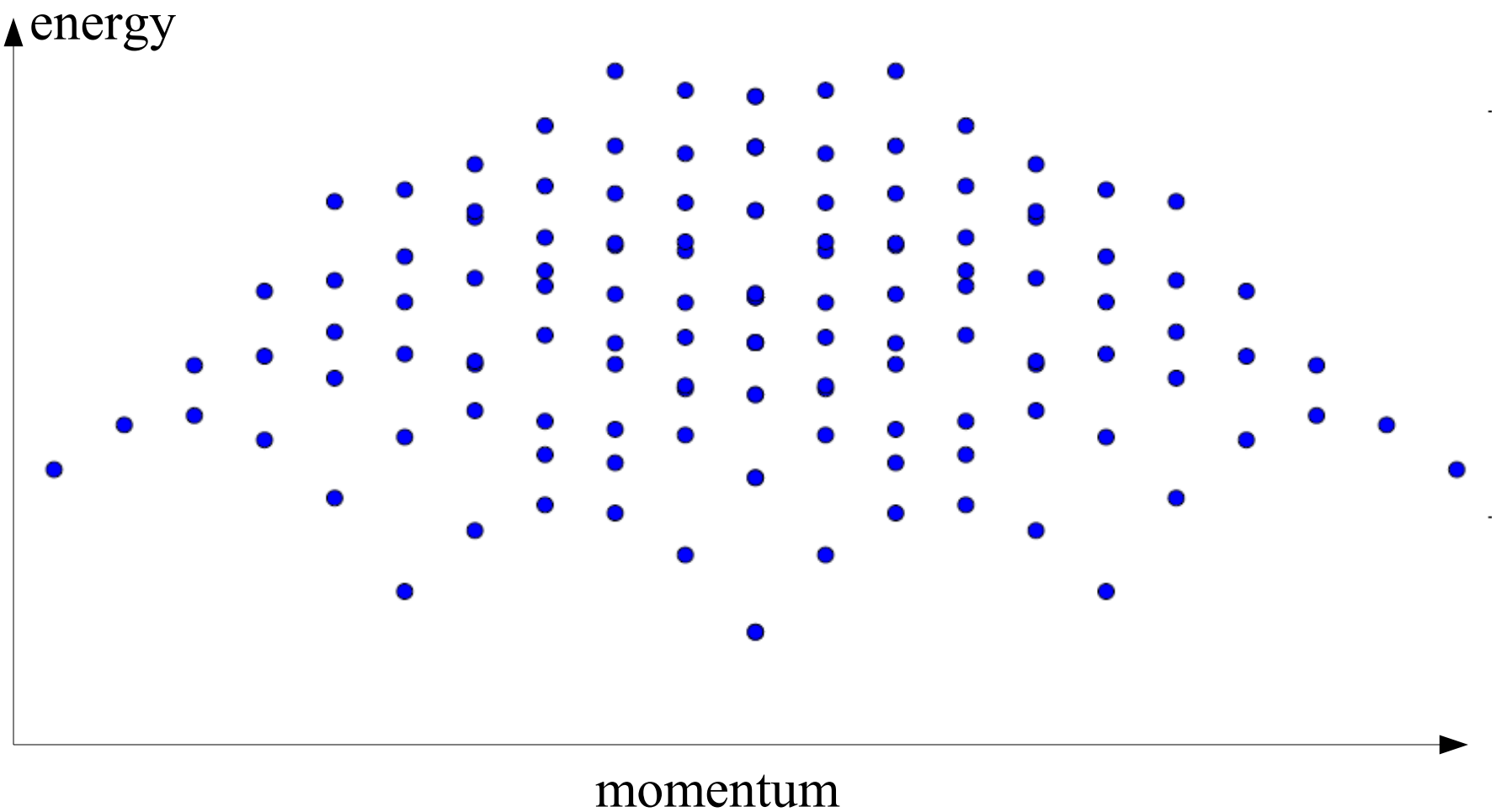


$$\mathcal{L}_{ch} = \frac{1}{2\pi}\partial_x\phi(\partial_t + v_c\partial_x\phi)$$

Edge states



Edge states



Luttinger exponents

With a parabolic potential it is possible to measure
The Luttinger K of the sum of the two chiral modes.

K=1/m for filling factor 1/m of simple Laughlin liquids.

Pfaffian state

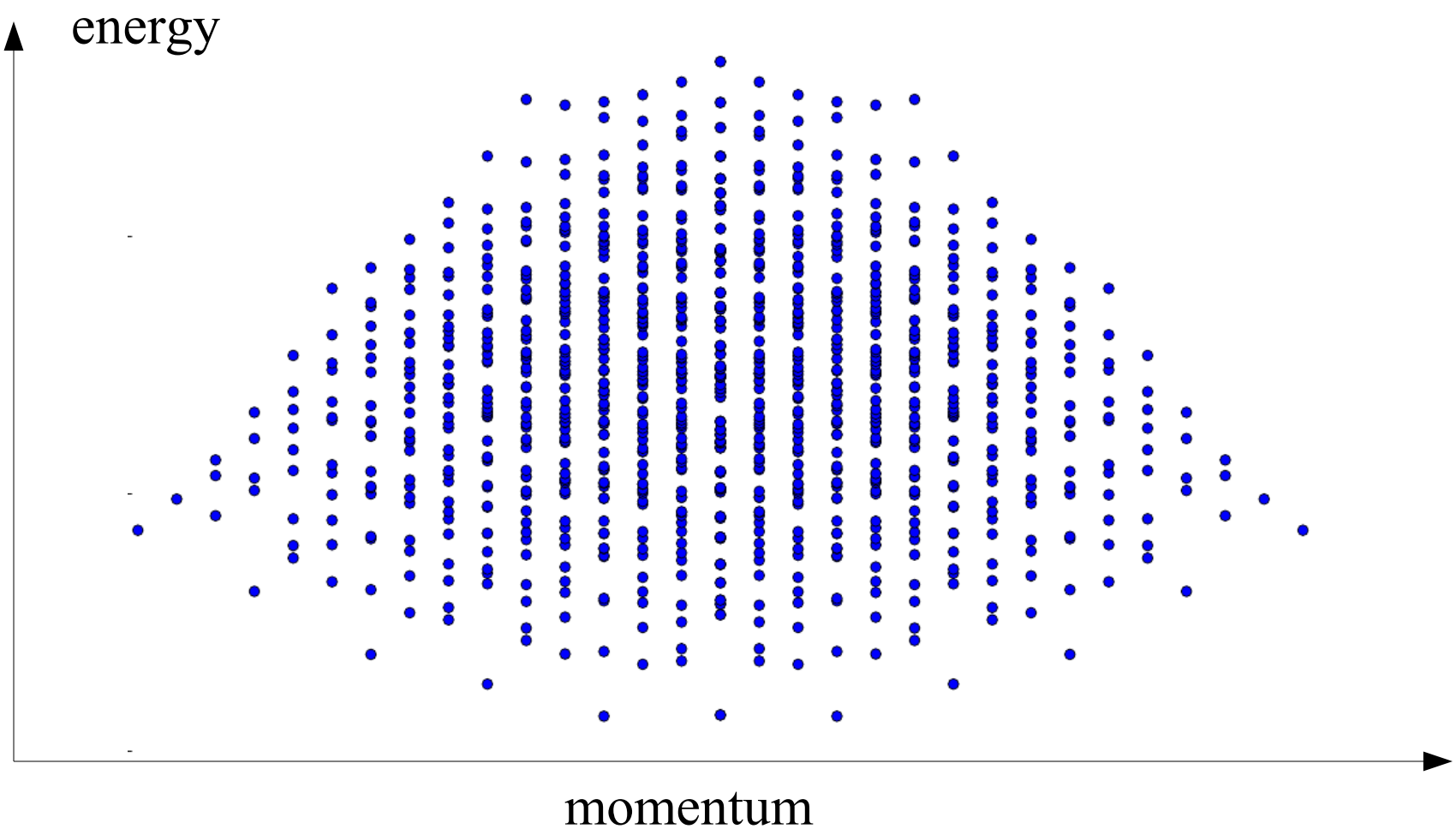
$$\Psi_{Moore/Read} = \prod_{j<k}(z_j - z_k)^2 \prod_j e^{-|z_j|^2/4} \cdot \mathrm{Pf}\left(\frac{1}{z_j - z_k}\right)$$

$$\mathrm{Pf}\left(\frac{1}{z_j - z_k}\right) = \mathcal{A}\left(\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots\right)$$

$$\Psi_{Moore/Read} = \Psi^{m=2}_{Laughlin} \times \Psi^{p_x+ip_y}_{BCS}$$

$$\Psi^{p_x+ip_y}_{BCS} = \mathrm{Pf}\left(\frac{1}{z_j - z_k}\right)$$

The Pfaffian edge



Conclusions and outlook

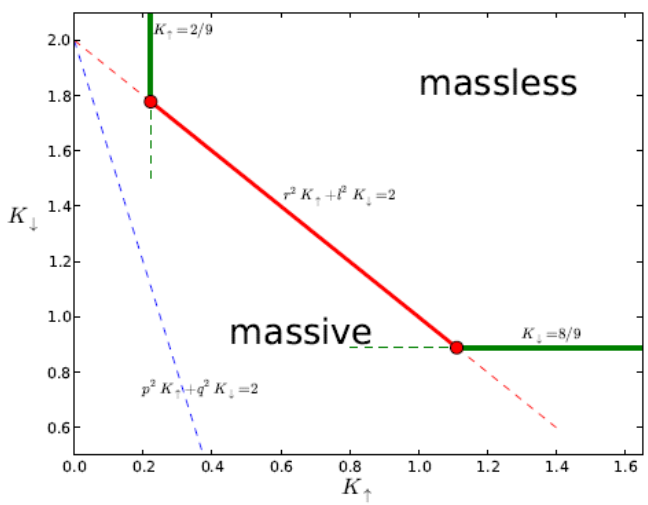
- Multiple partially gapped phases possible in density- and mass-imbalanced mixtures.
- (Quasi-)long-range ordering of several-particle composites
- Commensurability effects (lattice not required)
- D > 1 ?

FQHE states and their edge modes: an interesting goal

Manipulate edge modes, tunneling between droplets
Of atoms and of edge excitations, solve old puzzles
Of solid state world

Lattice effects, power counting

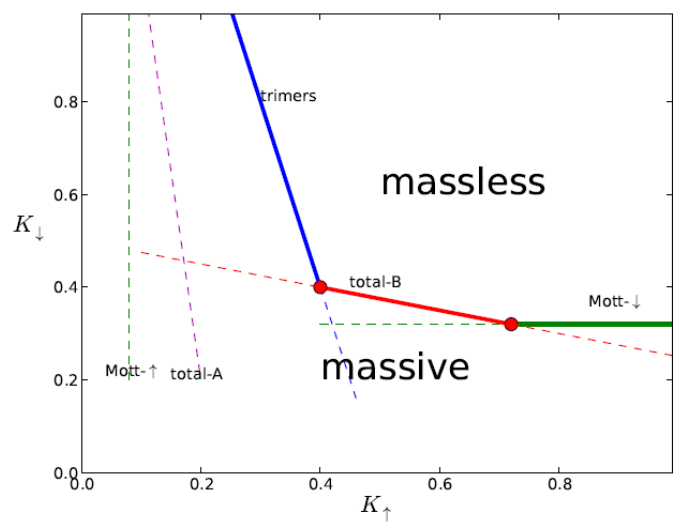
$$n_\uparrow = 1/3,\ n_\downarrow = 2/3$$



Power counting:

$$p^2 K_\uparrow + q^2 K_\downarrow \leqslant 2\ ,$$
$$r^2 K_\uparrow + l^2 K_\downarrow \leqslant 2\ ,$$
$$(pl + qr)^2 K_\downarrow \leqslant 2p^2\ ,$$
$$(pl + qr)^2 K_\uparrow \leqslant 2q^2\ ,$$

$$n_\uparrow = 1/5,\ n_\downarrow = 2/5$$



A: $l = 1$, $r = 3$
B: $l = 2$, $r = 1$

A microscopic example: $p > q = 1$

✵species: free fermions: $K_\downarrow = 1$

✸species: dipolar bosons, a Luttinger liquid with

$K_\uparrow \propto n_\uparrow^{-1/2}$ as $n_\uparrow \to \infty$ (Citro et al., 2007)

Take a majority of light non-interacting fermions and a minority of heavy dipolar bosons:

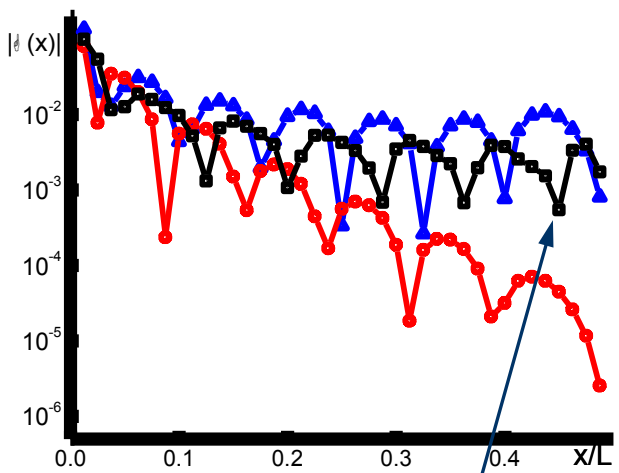
$$m_\uparrow = p\, m_\downarrow \qquad n_\uparrow \propto p^4$$
$$n_\uparrow = n_\downarrow / p$$

Switch on the coupling:

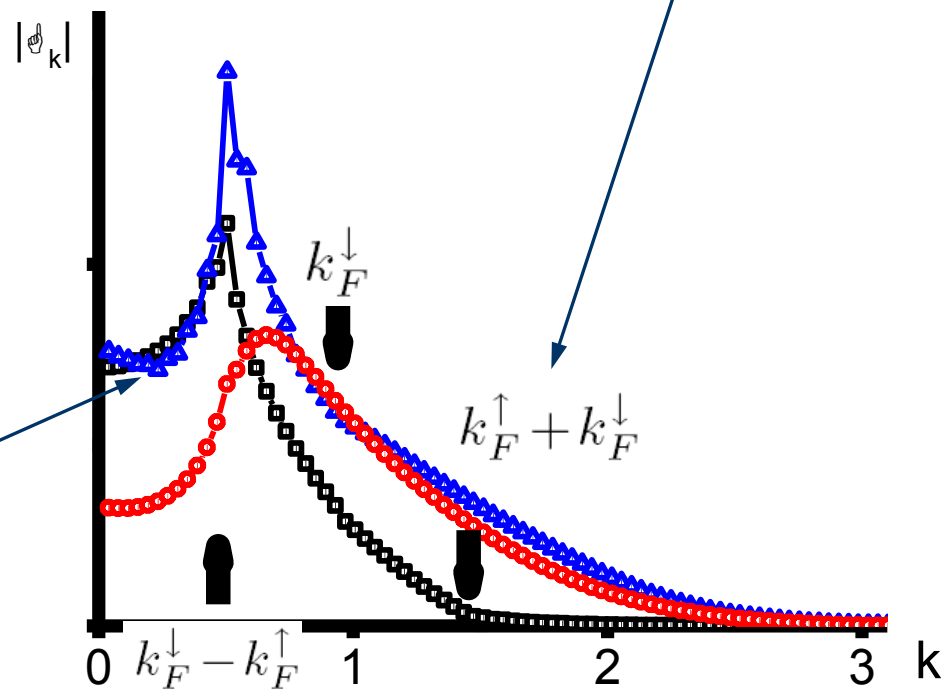
$$K_a = 1 + g \frac{K_\uparrow}{v_\uparrow} \frac{p}{4q} + O(g^2)$$

I. e.: $g < 0$ (an infinitesimal attraction) opens the gap.

The asymmetric Hubbard model, cont'd



Broadening of the momentum distribution is insensitive to the commensurability



long-range behavior is the same for

- equal masses
- unequal masses, incommensurate densities

The effects of higher harmonics, cont'd

$$\mathcal{H}_{\text{free}} = \mathcal{H}_0(\phi_\uparrow) + \mathcal{H}_0(\phi_\downarrow)$$

$$\mathcal{H}_0(\phi_\sigma) = \frac{v_\sigma}{2\pi} \int dx \left[K_\sigma (\pi \Pi_\sigma)^2 + K_\sigma^{-1} \left(\partial_x \phi_\sigma\right)^2 \right]$$

$$\mathcal{H}_1 = g \int dx \, (\partial_x \phi_\uparrow)(\partial_x \phi_\downarrow)$$

Renormalization group analysis (Penc and Sólyom, 1990) :
cos(…) is either relevant or irrelevant in the RG sence.

- cos(…) is irrelevant ➡ 1D FFLO phase
- cos(…) is relevant ➡ one of the eigenmodes acquires a mass.

Polarized Fermi superfluids : FFLO state

Fulde and Ferrell, 1964

Larkin and Ovchinnikov, 1965.

$$\Delta(x) \propto \cos Qx$$

$$Q = |k_F^{\uparrow} - k_F^{\downarrow}|$$

No direct experimental evidence yet:

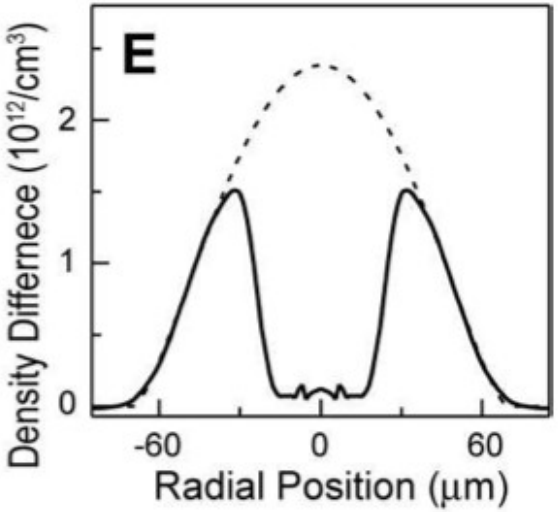
Partridge et al., 2006 (Rice)

Zwierlein et al., 2006 (MIT)

Cold gases with external confinement in 3D:

phase separation!

3D polarized gases are not superfluid…



$$\mathcal{H}_h = \sum_{s,s'>0} G_{ss'} \int dx \, \cos\left[2(sk_F^{\uparrow} - s'k_F^{\downarrow})x - 2(s\phi_{\uparrow} - s'\phi_{\downarrow})\right]$$
$$+ \sum_{s,s'>0} \tilde{G}_{ss'} \int dx \, \cos\left[2(sk_F^{\uparrow} + s'k_F^{\downarrow})x - 2(s\phi_{\uparrow} + s'\phi_{\downarrow})\right]$$

Consider a special case:

$$p\, n_{\uparrow} - q\, n_{\downarrow} = 0 \qquad n_{\uparrow} = q/(pl + qr)$$
$$r\, n_{\uparrow} + l\, n_{\downarrow} = 1 \qquad n_{\downarrow} = p/(pl + qr)$$

$$\mathcal{H}_h = G_1 \int dx \, \cos 2(p\,\phi_{\uparrow} - q\,\phi_{\downarrow})$$ as before

$$+G_2 \int dx \, \cos 2(r\,\phi_{\uparrow} + l\,\phi_{\downarrow})$$

$$+G_3 \int dx \, \cos 2(l + qr/p)\,\phi_{\downarrow}$$ Mott-❈

$$+G_4 \int dx \, \cos 2(r + pl/q)\,\phi_{\uparrow} \, .$$ Mott-✸

$$p^2 K_{\uparrow} + q^2 K_{\downarrow} \leqslant 2 \, ,$$
$$r^2 K_{\uparrow} + l^2 K_{\downarrow} \leqslant 2 \, ,$$
$$(pl + qr)^2 K_{\downarrow} \leqslant 2p^2 \, ,$$
$$(pl + qr)^2 K_{\uparrow} \leqslant 2q^2 \, ,$$

One dimension: "FFLO" state

In 1D, there's no true long-range order… But algebraic ordering is possible:

$$\left\langle \Psi_\uparrow \Psi_\downarrow(0)\, \Psi_\downarrow^\dagger \Psi_\uparrow^\dagger(x) \right\rangle \sim \cos|k_F^\uparrow - k_F^\downarrow| x \left(\frac{1}{x}\right)^\alpha$$

Bethe ansatz (integrable models):

Orso, 2007

Hu, Liu, and Drummond, 2007

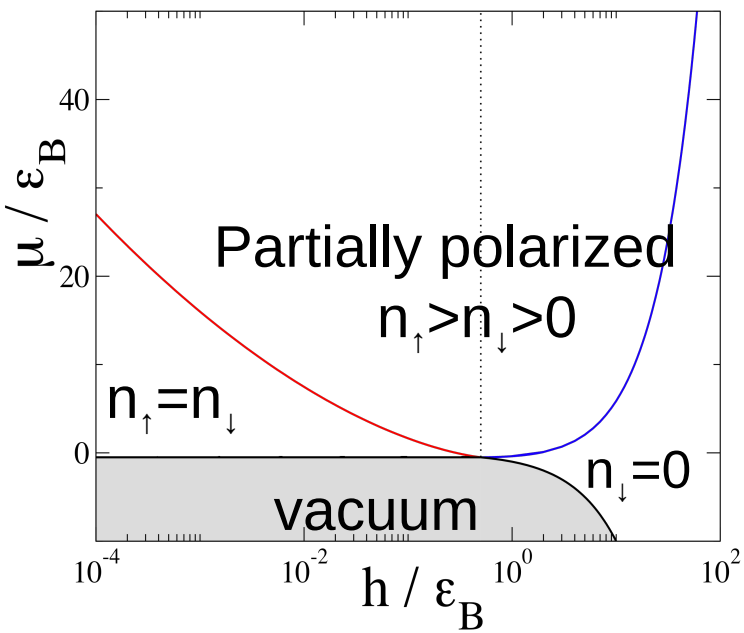
Numerics (DMRG & QMC):

Feiguin and Heidrich-Meisner, 2007

Batrouni et al., 2008

Rizzi et al., 2008

Experiment: R. Hulet et al., 2009



Different velocities

$$\mathcal{H}_{\text{free}} = \mathcal{H}_0(\phi_\uparrow) + \mathcal{H}_0(\phi_\downarrow)$$

$$\mathcal{H}_0(\phi_\sigma) = \frac{v_\sigma}{2\pi} \int dx \left[K_\sigma (\pi \Pi_\sigma)^2 + K_\sigma^{-1} \left(\partial_x \phi_\sigma\right)^2 \right]$$

$$\mathcal{H}_1 = g \int dx \, (\partial_x \phi_\uparrow)(\partial_x \phi_\downarrow)$$

Can diagonalize the bilinear part: new fields, $\phi_{1,2}$

with some velocities and Luttinger parameters: $v_{1,2} \quad K_{1,2}$