Bogoliubov theory of disordered Bose-Einstein condensates

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Junior Invited Speaker
Lyon BEC 2012
`http://lyonbec2012.ens-lyon.fr/`

(slightly amended version)

Summary

We tackle the interplay of interaction, disorder, and Bose statistics—a long standing problem known as the ”dirty boson problem”. Concretely, we present a *Bogoliubov theory* for disordered Bose-Einstein condensates, i.e., the bosonic field operator is split into the (mean field) condensate and (quantum) fluctuations. The mean-field part consists in solving the Gross-Pitaevskii equation to obtain the condensate wave function, which is deformed by the disorder potential. The deformed condensate, in turn, determines the Hamiltonian for the quantum fluctuations.
Diagonalizing this Bogoliubov Hamiltonian is a difficult task. As it is not desirable anyway to solve the problem for a particular realization of disorder, we resort to *disorder perturbation theory* in terms of Green functions to compute the mean free path of Bogoliubov excitations and the disorder-averaged sound velocity.
Furthermore, the Bogoliubov theory is used to count the number of particles that are excited out of the condensate, even at zero temperature. This *depletion of the condensate* is shown to remain small in presence of disorder, which validates a posteriori the Bogoliubov ansatz.

References:

C. Gaul & C.A. Müller, Phys. Rev. A, 83, 063629 (2011)

C.A. Müller & C. Gaul, arXiv:1202.3489 (2012)

Bogoliubov theory of disordered Bose-Einstein condensates

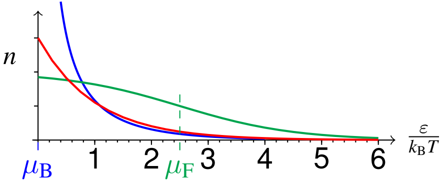
- Bose statistics + Interaction + Disorder
  $\rightarrow$ “Dirty Boson Problem”
- How is Bose-Einstein condensation affected by disorder?
  - How to define the condensate in presence of inhomogeneity?
  - Fraction of non-condensed particles
- How are the elementary excitations affected?

Bose statistics

- Classical:

- Bosons: indistinguishable, symmetric wf.: $\hat{a}_1^\dagger \hat{a}_2^\dagger \left|0\right\rangle = \hat{a}_1^\dagger \hat{a}_2^\dagger \left|0\right\rangle$
- Indistinguishable bosons tend to cluster

- $n_\mathrm{B}(\varepsilon) = \frac{1}{\exp\left(\frac{\varepsilon - \mu_\mathrm{B}}{k_\mathrm{B} T}\right) - 1}$



- Fermi: $n_\mathrm{F}(\varepsilon) = \frac{1}{\exp\left(\frac{\varepsilon - \mu_\mathrm{F}}{k_\mathrm{B} T}\right) + 1}$
- Boltzmann: $n(\varepsilon) \propto e^{-\frac{\varepsilon}{k_\mathrm{B} T}}$

- n_B diverges for $\varepsilon \to \mu_\mathrm{B} \Rightarrow$ "Bose-Einstein" condensation (BEC)
  If: thermal de-Broglie wavelength $\sim$ (particle density)$^{1/d}$

Penrose-Onsager criterion

- Starting point: bosonic many-body Hamiltonian

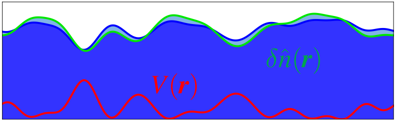
$$E[\hat{\Psi}, \hat{\Psi}^\dagger] = \int \mathrm{d}^d r\, \hat{\Psi}^\dagger(\boldsymbol{r}) \left[\frac{-\hbar^2}{2m} \nabla^2 + V(\boldsymbol{r}) + \frac{g}{2} \hat{\Psi}^\dagger(\boldsymbol{r}) \hat{\Psi}(\boldsymbol{r}) - \mu \right] \hat{\Psi}(\boldsymbol{r})$$

- One-body denity matrix (OBDM): $\langle \hat{\Psi}^\dagger(\boldsymbol{r}) \hat{\Psi}(\boldsymbol{r}') \rangle$
- BEC: many particles occupy condensate orbital
- Penrose & Onsager (1956): $\int \mathrm{d}^d r' \underbrace{\langle \hat{\Psi}^\dagger(\boldsymbol{r}) \hat{\Psi}(\boldsymbol{r}') \rangle}_{\text{OBDM}} \Phi(\boldsymbol{r}') = N_\mathrm{c} \Phi(\boldsymbol{r})$
  - Condensate $\Phi(\boldsymbol{r})$
  - Number of condensed particles $N_\mathrm{c} \gg 1$

Mean field and Bogoliubov theory

Condensate and quantum fluctuations

$$\hat{\Psi}(\boldsymbol{r}) = \Phi(\boldsymbol{r}) + \delta\hat{\psi}(\boldsymbol{r}, t)$$



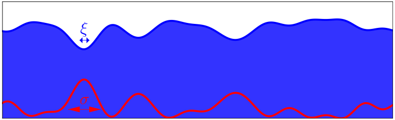
Meanfield: Minimize $E[\Phi] \to$ Gross-Pitaevskii equation

$$\left[\frac{-\hbar^2}{2m}\nabla^2 + g|\Phi(\boldsymbol{r})|^2 + V(\boldsymbol{r}) - \mu\right]\Phi(\boldsymbol{r}) = 0$$

Weak V: $\Phi(\boldsymbol{r}) = \sqrt{n_c} + \Phi^{(1)}(\boldsymbol{r})$

$\Phi^{(1)}_{\boldsymbol{k}} = \frac{-V_k}{\varepsilon^0_k + 2gn_c}$

$n_c = \overline{|\Phi(\boldsymbol{r})|^2}$, $\xi^2 = \hbar^2/(2mgn_c)$



Effective Hamiltonian for quantum fluctuations

$$E[\Phi + \delta\hat{\psi}] \approx E[\Phi] + \underbrace{\frac{1}{2}\int \mathrm{d}^d r\, \mathrm{d}^d r' \left(\delta\hat{\psi}^\dagger(\boldsymbol{r}'), \delta\hat{\psi}(\boldsymbol{r}')\right) \mathcal{H}(\boldsymbol{r}', \boldsymbol{r}) \begin{pmatrix} \delta\hat{\psi}(\boldsymbol{r}) \\ \delta\hat{\psi}^\dagger(\boldsymbol{r}) \end{pmatrix}}_{\hat{H}}$$

$$\mathcal{H} = \delta(\boldsymbol{r}-\boldsymbol{r}') \left\{ \left[\frac{-\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r}) - \mu \right] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + g \begin{pmatrix} |\Phi(\boldsymbol{r})|^2 & \frac{1}{2}\Phi(\boldsymbol{r})^2 \\ \frac{1}{2}\Phi^*(\boldsymbol{r})^2 & |\Phi(\boldsymbol{r})|^2 \end{pmatrix} \right\}$$

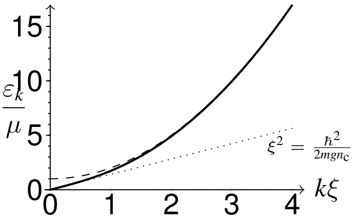
- In terms of density and phase: $\Phi(\boldsymbol{r}) + \hat{\psi}(\boldsymbol{r}) = e^{\mathrm{i}\delta\hat{\varphi}(\boldsymbol{r})}\sqrt{n_\mathrm{c} + \delta\hat{n}(\boldsymbol{r})}$
- Fourier- & Bogoliubov trafo: “bogolons”

$$\hat{\gamma}_{\boldsymbol{k}} = \delta\hat{n}_{\boldsymbol{k}}/(2a_k\sqrt{n_\mathrm{c}}) + \mathrm{i}a_k\sqrt{n_\mathrm{c}}\,\delta\hat{\varphi}_{\boldsymbol{k}} \qquad a_k = \sqrt{\varepsilon_k^0/\varepsilon_k}$$

$$\hat{H} = \sum_{\boldsymbol{k}} \varepsilon_k \hat{\Gamma}_{\boldsymbol{k}}^\dagger \hat{\Gamma}_{\boldsymbol{k}} + \sum_{\boldsymbol{k},\boldsymbol{k}'} \hat{\Gamma}_{\boldsymbol{k}}^\dagger \mathcal{V}_{\boldsymbol{k}\boldsymbol{k}'} \hat{\Gamma}_{\boldsymbol{k}'}, \qquad \hat{\Gamma}_{\boldsymbol{k}} = \begin{pmatrix} \hat{\gamma}_{\boldsymbol{k}} \\ \hat{\gamma}_{-\boldsymbol{k}}^\dagger \end{pmatrix}$$

Homogeneous Bogoliubov problem

$$\hat{H}^{(0)} = \sum_{\boldsymbol{k}} \varepsilon_k \hat{\gamma}^\dagger_{\boldsymbol{k}} \hat{\gamma}_{\boldsymbol{k}}$$



- Bogoliubov dispersion relation

$$\varepsilon_k = \sqrt{\varepsilon_k^0(2gn_c + \varepsilon_k^0)}, \quad \varepsilon_k^0 = \frac{\hbar^2 k^2}{2m}$$

- Condensate depletion

$$\delta n^{(0)} = L^{-d} \sum_{\boldsymbol{k}} \langle \delta\hat{\psi}^\dagger_{\boldsymbol{k}} \delta\hat{\psi}_{\boldsymbol{k}} \rangle = L^{-d} \sum_{\boldsymbol{k}} v_k^2 \overset{(3D)}{=} \frac{1}{6\sqrt{2}\pi^2} \xi^{-3} \propto \xi^{-d}$$

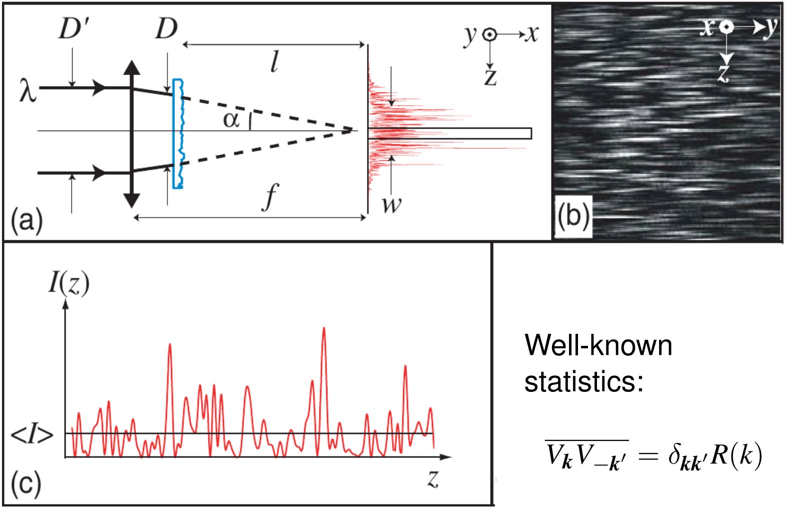
$$\delta\hat{\psi}_{\boldsymbol{k}} = u_k \hat{\gamma}_{\boldsymbol{k}} + v_k \hat{\gamma}^\dagger_{-\boldsymbol{k}}$$

- Relative depletion $\frac{\delta n^{(0)}}{n_c} \overset{(3D)}{=} \frac{8}{3\sqrt{\pi}} \sqrt{n a_s^3}$

↑ dilute-gas parameter

[Lee, Huang & Yang (1957)]

Laser-speckle disorder potential



Well-known statistics:

$$\overline{V_{\mathbf{k}} V_{-\mathbf{k}'}} = \delta_{\mathbf{k}\mathbf{k}'} R(k)$$

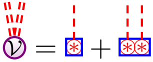
[Clément et al., New J. Phys., **8**, 165 (2006)]

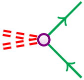
Bogolons in a disordered medium

- Hamiltonian

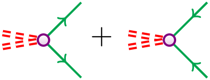
"Bogoliubov-Nambu spinor"

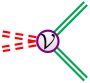
$$\hat{H} = \sum_{\boldsymbol{k}} \varepsilon_k \hat{\Gamma}^\dagger_{\boldsymbol{k}} \hat{\Gamma}_{\boldsymbol{k}} + \sum_{\boldsymbol{k},\boldsymbol{k}'} \hat{\Gamma}^\dagger_{\boldsymbol{k}} \mathcal{V}_{\boldsymbol{k}\boldsymbol{k}'} \hat{\Gamma}_{\boldsymbol{k}'}, \qquad \hat{\Gamma}_{\boldsymbol{k}} = \begin{pmatrix} \hat{\gamma}_{\boldsymbol{k}} \\ \hat{\gamma}^\dagger_{-\boldsymbol{k}} \end{pmatrix}$$

- Vertex $\mathcal{V} = \begin{pmatrix} W & Y \\ Y & W \end{pmatrix} =$  $+ \dots$

- $W^{(1)}_{\boldsymbol{k}\boldsymbol{p}} = \frac{g n_c}{\sqrt{N_c}} \xi^2 \left[\frac{\boldsymbol{k}\cdot\boldsymbol{p}}{a_k a_p} - a_k a_p (k^2 + p^2 - \boldsymbol{k}\cdot\boldsymbol{p}) \right] \underbrace{\Phi^{(1)}_{\boldsymbol{k}-\boldsymbol{p}}}_{\propto V_{\boldsymbol{k}-\boldsymbol{p}}}$ 

- Anomalous scattering

$$Y_{\boldsymbol{k}',-\boldsymbol{k}} \hat{\gamma}^\dagger_{\boldsymbol{k}'} \hat{\gamma}^\dagger_{\boldsymbol{k}} + Y_{-\boldsymbol{k}',\boldsymbol{k}} \hat{\gamma}_{\boldsymbol{k}'} \hat{\gamma}_{\boldsymbol{k}} =$$ 

- Combined vertex 

Disorder-averaged effective medium

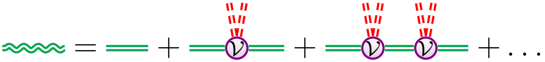
How do Booliubov quasi-particles travel on average through the disordered medium?

- Matrix-valued (retarded) Green function

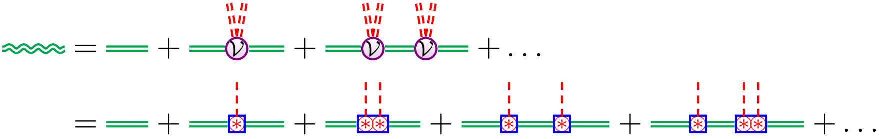
$$\mathcal{G}_{\boldsymbol{k}\boldsymbol{k}'}(t) = \frac{\Theta(t)}{i\hbar} \left\langle \left[\hat{\Gamma}_{\boldsymbol{k}}(t), \hat{\Gamma}^{\dagger}_{\boldsymbol{k}'}(0) \right] \right\rangle ,$$

Contains dispersion relation: $[\mathcal{G}_0(\boldsymbol{k},\omega)]_{11} = [\hbar\omega - \varepsilon_k + i0^+]^{-1}$

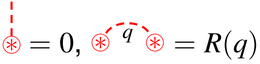
- Expansion in terms scattering vertex $\mathcal{V} =$  and $\mathcal{G}_0 =$ :

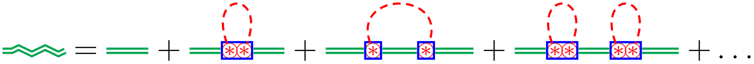
$\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_0 \mathcal{V} \mathcal{G} \quad \Leftrightarrow$ 

Computing the disorder-averaged Green function



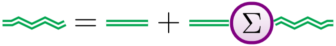
Disorder average:

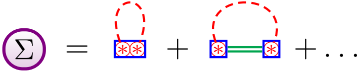




(reducible)

Dyson equation:

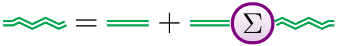




(irreducible)

Meaning of the self energy

- Disorder-averaged Green function



$$\overline{\mathcal{G}} = \mathcal{G}_0 + \mathcal{G}_0 \Sigma \overline{\mathcal{G}}$$

$$\overline{\mathcal{G}} = \left[\mathcal{G}_0^{-1} - \Sigma\right]^{-1} = \left[\hbar\omega - \varepsilon_k - \Sigma + i0^+\right]^{-1}$$

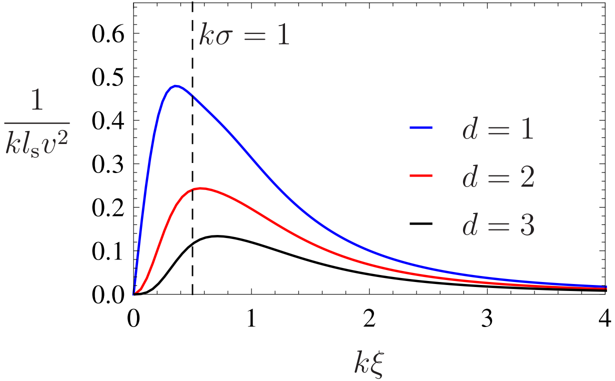
- Renormalized dispersion relation

$$\hbar\omega = \varepsilon_k + \Sigma_{11}^{(2)}(k,\omega)$$

- $\mathrm{Im}\Sigma \rightarrow$ finite mean free path

Mean free path

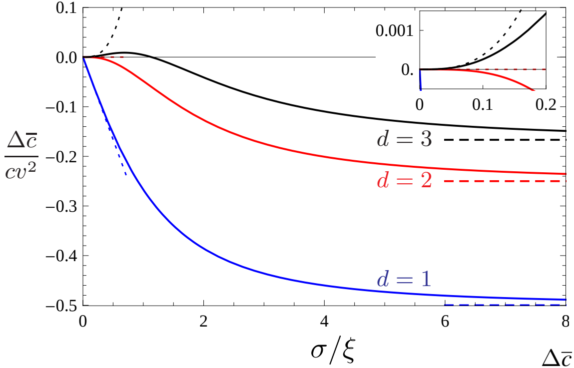
- Disorder $\overline{V_{\boldsymbol{q}} V_{-\boldsymbol{q}'}} = L^{-d} \delta_{\boldsymbol{q}\boldsymbol{q}'} \underbrace{(2\pi)^{\frac{d}{2}} \sigma^d (v g n_{\mathrm{c}})^2 e^{-\frac{q^2 \sigma^2}{2}}}_{R(q)}$
- Finite mean free path $(k l_{\mathrm{s}})^{-1} \propto |\mathrm{Im}\Sigma|$



- Related to localization of Bogoliubov quasiparticles
  [Lugan et al. PRA (2011)]

Disorder-renormalized speed of sound

ReΣ renormalizes sound velocity



$$v_\delta^2 = R(0)/(gn_c\xi^d)$$

$\Delta\bar{c}/c$	$\sigma \gg \xi$	$\sigma \ll \xi$
$d=1$	$-v^2/2$	$-\frac{3}{16\sqrt{2}}v_\delta^2$
$d=2$	$-v^2/4$	0
$d=3$	$-v^2/6$	$+\frac{5}{48\sqrt{2}\pi}v_\delta^2$ *

* [Giorgini et al., PRB 1994]

Momentum distribution of fluctuations

- To compute: $\delta n_{\boldsymbol{k}} = \langle \delta\hat{\Psi}^{\dagger}_{\boldsymbol{k}} \delta\hat{\Psi}_{\boldsymbol{k}} \rangle$
- We have: Hamiltonian for $\hat{\gamma}_{\boldsymbol{k}} = \delta\hat{n}_{\boldsymbol{k}}/(2a_k\sqrt{n_c}) + \mathrm{i}a_k\sqrt{n_c}\delta\hat{\varphi}_{\boldsymbol{k}}$

$$\delta\hat{\psi}(\boldsymbol{r}) = \delta\hat{n}(\boldsymbol{r})/[2\Phi(\boldsymbol{r})] + \mathrm{i}\Phi(\boldsymbol{r})\delta\hat{\varphi}(\boldsymbol{r})$$

- Transformation $\delta\hat{\Psi}_{\boldsymbol{k}} = \sum_{\boldsymbol{p}} \left(u_{\boldsymbol{kp}}\hat{\gamma}_{\boldsymbol{p}} - v_{\boldsymbol{kp}}\hat{\gamma}^{\dagger}_{-\boldsymbol{p}} \right)$, with

$$u_{\boldsymbol{kp}} = \frac{1}{2\sqrt{N_c}} \left[a_p^{-1}\Phi_{\boldsymbol{k}-\boldsymbol{p}} + a_p\check{\Phi}_{\boldsymbol{k}-\boldsymbol{p}} \right], \qquad \check{\Phi}_{\boldsymbol{k}} = [n_c/\Phi(\boldsymbol{r})]_{\boldsymbol{k}}$$

$$v_{\boldsymbol{kp}} = \frac{1}{2\sqrt{N_c}} \left[a_p^{-1}\Phi_{\boldsymbol{k}-\boldsymbol{p}} - a_p\check{\Phi}_{\boldsymbol{k}-\boldsymbol{p}} \right]$$

$$\delta n_{\boldsymbol{k}} = \sum_{\boldsymbol{p},\boldsymbol{p}'} \Big\{ \delta_{\boldsymbol{pp}'}|v_{\boldsymbol{kp}}|^2 + \left(u^*_{\boldsymbol{kp}}u_{\boldsymbol{kp}'} + v^*_{\boldsymbol{kp}}v_{\boldsymbol{kp}'}\right)\langle\hat{\gamma}^{\dagger}_{\boldsymbol{p}}\hat{\gamma}_{\boldsymbol{p}'}\rangle - \left(u^*_{\boldsymbol{kp}}v_{\boldsymbol{kp}'}\langle\hat{\gamma}^{\dagger}_{\boldsymbol{p}}\hat{\gamma}^{\dagger}_{-\boldsymbol{p}'}\rangle + c.c.\right) \Big\}$$

at $T = 0$: only due to inhomogeneity $V(\boldsymbol{r})$

homogeneous quantum depletion

Momentum distribution of fluctuations

Pick second-order terms of

$$\delta n_{\boldsymbol{k}} = \sum_{\boldsymbol{p},\boldsymbol{p}'} \Big\{ \delta_{\boldsymbol{p}\boldsymbol{p}'} |v_{\boldsymbol{k}\boldsymbol{p}}|^2 + \big(u^*_{\boldsymbol{k}\boldsymbol{p}} u_{\boldsymbol{k}\boldsymbol{p}'} + v^*_{\boldsymbol{k}\boldsymbol{p}} v_{\boldsymbol{k}\boldsymbol{p}'}\big) \langle \hat{\gamma}^\dagger_{\boldsymbol{p}} \hat{\gamma}_{\boldsymbol{p}'} \rangle - \big(u^*_{\boldsymbol{k}\boldsymbol{p}} v_{\boldsymbol{k}\boldsymbol{p}'} \langle \hat{\gamma}^\dagger_{\boldsymbol{p}} \hat{\gamma}^\dagger_{-\boldsymbol{p}'} \rangle + c.c.\big) \Big\}$$

- $\langle \hat{\gamma}^\dagger_{\boldsymbol{p}} \hat{\gamma}_{\boldsymbol{p}'} \rangle = \langle \hat{\gamma}^\dagger_{\boldsymbol{p}} \hat{\gamma}_{\boldsymbol{p}'} \rangle^{(0)} + \langle \hat{\gamma}^\dagger_{\boldsymbol{p}} \hat{\gamma}_{\boldsymbol{p}'} \rangle^{(1)} + \langle \hat{\gamma}^\dagger_{\boldsymbol{p}} \hat{\gamma}_{\boldsymbol{p}'} \rangle^{(2)} + \dots$
- $u_{\boldsymbol{k}\boldsymbol{p}} = u^{(0)}_{\boldsymbol{k}\boldsymbol{p}} + u^{(1)}_{\boldsymbol{k}\boldsymbol{p}} + u^{(2)}_{\boldsymbol{k}\boldsymbol{p}} + \dots$

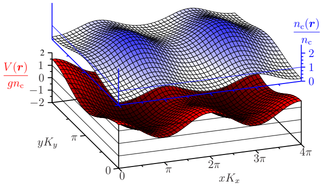
$$\Rightarrow \quad \delta n^{(2)}_{\boldsymbol{k}} = \sum_{\boldsymbol{p}} M^{(2)}_{\boldsymbol{k}\boldsymbol{p}} |V_{\boldsymbol{k}-\boldsymbol{p}}|^2$$

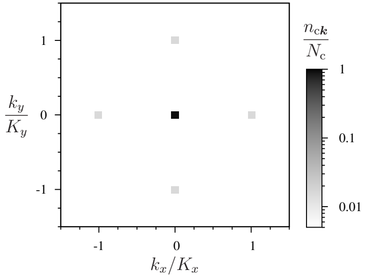
↑ a "monstrous" envelope function

Momentum distribution in a 2D lattice

$$V(\boldsymbol{r}) = \sum_j V_j \cos(\boldsymbol{K}_j \cdot \boldsymbol{r})$$

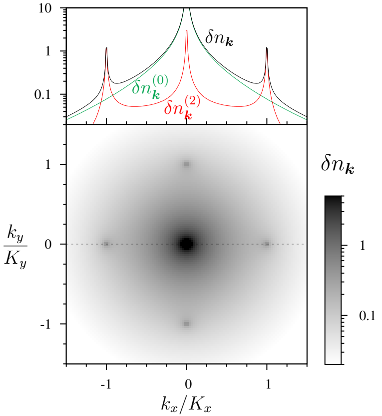
Condensate deformation $|\Phi_k|^2$





Quantum fluctuations $\delta n_{\boldsymbol{k}}$

- "Quantum depletion" $\delta n^{(0)}$
- "Potential depletion" $\delta n^{(2)}$



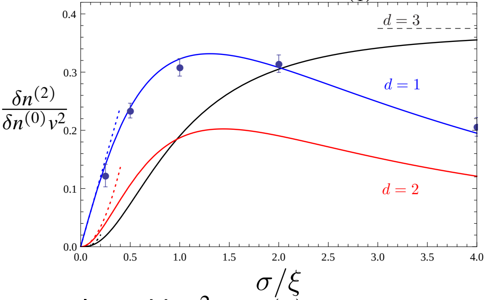
Condensate depletion due to Gaussian disorder

$$\delta n^{(2)} = L^{-d} \sum_{\boldsymbol{k}\boldsymbol{p}} M^{(2)}_{\boldsymbol{k}\boldsymbol{p}} \overline{|V_{\boldsymbol{k}-\boldsymbol{p}}|^2}$$

$$\overline{|V_{\boldsymbol{q}}|^2} = L^{-d} \underbrace{(2\pi)^{\frac{d}{2}} \sigma^d (v g n_c)^2 e^{-\frac{q^2\sigma^2}{2}}}_{R(q)}$$

$\frac{\delta n^{(2)}}{\delta n^{(0)}}$	$\sigma \gg \xi$	$\sigma \ll \xi$
$d=1$	$-\frac{1}{8}v^2$	$0.245 v_\delta^2$
$d=2$	0	$0.135 v_\delta^2$
$d=3$	$\frac{3}{8}v^2$	$0.160 v_\delta^2$

$$v_\delta^2 = R(0)/(g n_c \xi^d)$$



- $\sigma \ll \xi$: Depletion correction scales with $v_\delta^2 \propto R(0)$
- $\sigma \gg \xi$: Local density approximation

$$\delta n^{(0)} \propto \xi^{-d} \propto (g n_c)^{d/2} = (\mu - V)^{d/2} \approx \mu \left[1 - \frac{d}{2}v + \frac{v^2}{8} d(d-2)\right]$$

$$\Rightarrow \frac{\delta n^{(2)}_{\mathrm{TF}}}{\delta n^{(0)}} = \frac{d(d-2)v^2}{8}$$

Take-home messages

- ✓ Hamiltonian for quantum excitations on top of deformed condensate
- ✓ Diagrammatic disorder perturbation theory
  - Mean free path
  - Renormalized speed of sound
- ✓ Calculation of the potential-induced condensate depletion
  - Depletion remains small ⇒ validates Bogoliubov ansatz

References

C. Gaul and C. A. Müller, Phys. Rev. A, 83, 063629 (2011)
C. A. Müller and C. Gaul, arXiv:1202.3489 (2012)

Thanks!

Cord A. Müller

Funding: Moncloa Campus of International Excellence (UCM-UPM)